$V \to E \to D \to \sigma$

MidTerm examination

Time allocated: 80 minutes

This exam contains **3 questions** on two pages.
Only pen or pencil, eraser and calculator allowed. Everything else is supplied with the exam.
Remember to obtain a cheat sheet. If you use any equation from the cheat sheet, please indicate so at the appropriate place on your copy.

This maximum score for this exam is 25.
If you accumulate more than 25 marks, your score will be 25.

You may (or may not) find some of the following integrals useful:

$$\int \frac{dx}{(x^2+a^2)^{1/2}} = \ln(x+\sqrt{a^2+x^2}), \qquad \int \frac{x\,dx}{(x^2+a^2)^{1/2}} = \sqrt{a^2+x^2},$$

$$\int \frac{dx}{(x^2+a^2)} = \frac{1}{a}\arctan\left(\frac{x}{a}\right), \qquad \int \frac{x\,dx}{(x^2+a^2)} = \tfrac{1}{2}\ln(a^2+x^2)$$

$$\int \frac{dx}{(x^2+a^2)^{3/2}} = \frac{x}{a^2\sqrt{x^2+a^2}}, \qquad \int \frac{x\,dx}{(x^2+a^2)^{3/2}} = -\frac{1}{\sqrt{x^2+a^2}}$$

$$\int \frac{x^2\,dx}{(x^2+a^2)^{3/2}} = -\frac{x}{\sqrt{x^2+a^2}} + \ln\left(x+\sqrt{a^2+x^2}\right), \qquad \int \frac{x^2\,dx}{(x^2+a^2)^{1/2}} = \tfrac{1}{2}x\sqrt{x^2+a^2} - \tfrac{1}{2}a^2\ln\left(x+\sqrt{a^2+x^2}\right)$$

---

1. (10 points) A wire is placed along $\hat{x}$ between $(-L,0,0)$ and $(L,0,0)$. It is charged with a linear charge density $\lambda_\ell(x)$ that varies according

$$\lambda_\ell(x) = \lambda_0 \frac{x}{2L}, \tag{1}$$

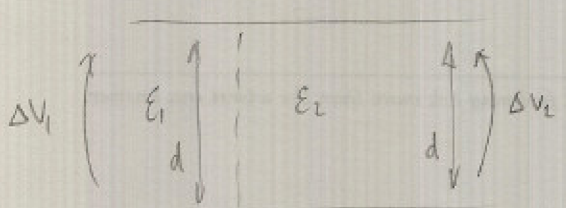
with λ_0 a constant.
Find the electric field at a point $(0, y_p, 0)$ with $y_p > 0$.

2. (15 points) (In this problem, r denotes the radial cylindrical coordinate.) A coaxial cable is constructed from an inner conducting wire of radius r_1 and a coaxial conducting hollow wire of inner radius $2r_1$ and outer radius r_3. The region $r_1 < r < \frac{3}{2}r_1$, is filled with a dielectric of relative permittivity $\varepsilon_r = 2$. If a constant surface charge distribution σ_s is placed on the inner cable:

a) (5 points) find the maximum value of $|\vec{E}|$ inside the coaxial cable,

b) (10 points) find the capacitance of this arrangement. (You can give a final answer, including all logs, unevaluated.)

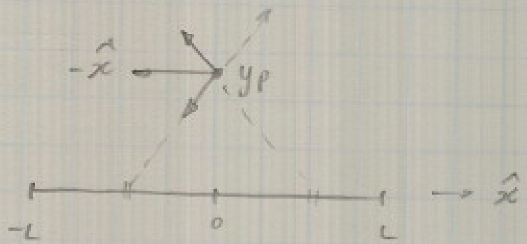
Please turn over



Since $\Delta V_1 = \Delta V_2$, then ~~$\Delta V_{12}=$~~

$E_1 = E_2 \to$ must argue that this condition holds everywhere

Q1.



resultant is in $-\hat{x}$ direction.

$$dE = \frac{dq}{4\pi\varepsilon_0} \cdot \frac{(y_p\hat{y} - x_s\hat{x})}{(x_s^2 + y_p^2)^{3/2}} \qquad dq = \frac{\lambda_0 x_s}{2L}\, dx_s$$

$$= \frac{\lambda_0 x_s}{8\pi L\varepsilon_0} \frac{(y_p\hat{y} - x_s\hat{x})}{(y_p^2 + x_p^2)^{3/2}}$$

$$E_x = -\int_{-L}^{L} \frac{\lambda_0\, x_0^2\, dx_s}{8\pi L\varepsilon_0\, (y_p^2 + x_s^2)^{3/2}} = \frac{-\lambda_0}{8\pi\varepsilon_0 L}\int_{-L}^{L} \frac{x_s^2\, dx_s}{(y_p^2 + x_s^2)^{3/2}}$$

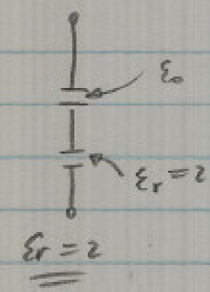
$$= \frac{-\lambda_0}{8\pi\varepsilon_0 L}\left[\frac{-x_s}{\sqrt{x_s^2 + y_p^2}} + \ln\left(x + \sqrt{y_p^2 + x_s^2}\right)\Big|_{-L}^{L}\right]$$

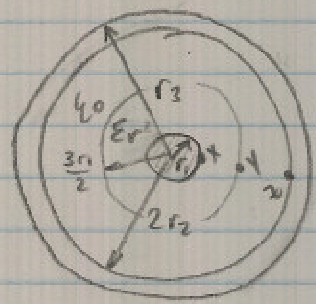
integral is given on exam sheet!

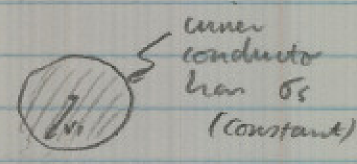
$$\boxed{\underline{\underline{E_x}} = \frac{1}{4\pi\varepsilon_0} \cdot \frac{\lambda_0}{2L}\left[\frac{2L}{\sqrt{L^2 + y_p^2}} - \ln\left(\frac{\sqrt{y_p^2 + L^2} + L}{\sqrt{y_p^2 + L^2} - L}\right)\right]}$$

Q2: Since the charge density on the inner conductor is given as constant, we may argue that the field is uniform and Gauss' Law applies.

(2)







(a) $|\vec{E}|_{max}$ inside cable:

GAUSS' LAW

$$\oint \vec{D} \cdot d\vec{s} = Q_{encl.}$$

$$\vec{D} = \varepsilon \vec{E} \qquad\qquad Q_{encl} = 2\pi r_1 l \sigma_s$$

variable radius

$$d\vec{s} = 2\pi r l$$

$$\vec{E} = \frac{\cancel{2\pi} r_1 \cancel{l} \sigma_s}{\varepsilon \cancel{2\pi} r \cancel{l}} = \frac{r_1 \sigma_s}{r \varepsilon} \hat{r}$$

$|\vec{E}|_{max}$ is where $r = r_1$, $\boxed{|\vec{E}_{max}| \quad \frac{\sigma_s}{\varepsilon_r}} = \left(\frac{\sigma_s}{2}\right)$

(b) $C = \frac{Q}{V}$ → this arrangement represents two series-connected capacitors.

Capacitor 1:

Gaussian ~~sphe~~ surface $\qquad r_1 < r < \frac{3r_1}{2}$

$$\oint \vec{D} \cdot d\vec{s} = Q_{encl}$$

$$\vec{D} = \varepsilon_r \varepsilon_0 \vec{E} \qquad\qquad Q_{encl} = 2\pi r_1 l \sigma_s$$

$$d\vec{s} = r \cdot 2\pi \cdot l$$

$$\vec{E} = \frac{2\pi r_1 l \sigma_s}{\varepsilon_r \cdot r \cdot 2\pi l} = \frac{r_1 \sigma_s}{\varepsilon_0 \varepsilon_r r}\hat{r}$$

$$\varepsilon_r = 2 \quad \therefore \quad \vec{E} = \frac{r_1 \sigma_s}{2\varepsilon_0 r}\hat{r}$$

$$V_{xy} = \frac{-r_1\sigma_s}{2\varepsilon_0}\int_{r_1}^{\frac{3r_1}{2}} \frac{dr}{r} = -\frac{r_1\sigma_s}{2\varepsilon_0} \ln r \Big|_{r_1}^{\frac{3r_1}{2}}$$

$$|V_1| = \frac{r_1\sigma_s}{2\varepsilon_0}\left[\ln\left(\frac{3r_1}{2}\right) - \ln(r_1)\right] = \frac{r_1\sigma_s}{2}\ln\left(\frac{3}{2}\right)$$

$$C_1 = \frac{Q_1}{|V_1|} = \frac{2\pi \cancel{r_1} l \cancel{\sigma_s}}{\frac{\cancel{r_1 \sigma_s}}{2\varepsilon_0}\ln\left(\frac{3}{2}\right)} = \frac{4\varepsilon_0 \pi l}{\ln\left(\frac{3}{2}\right)}$$

Capacitor 2: $\quad C_2 = \dfrac{Q_2}{|V_2|}$

$$Q_2 = 2\pi r_1 l \sigma_s$$

$$\oint \vec{D}\cdot d\vec{s} = Q_{enc} = 2\pi r_1 l \sigma_s$$

$$d\vec{s} = 2\pi r l$$

$$\vec{D} = \varepsilon \vec{E}$$

$$\vec{E} = \frac{\cancel{2\pi} r_1 \cancel{l} \sigma_s}{\varepsilon_0 \cancel{2\pi} r \cancel{l}}\hat{r}$$

$$\vec{E} = \frac{r_1\sigma_s}{\varepsilon_0 r}\hat{r}$$

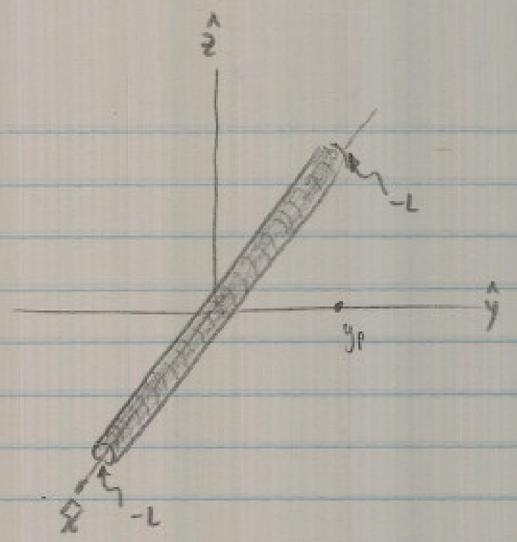
$$V_{yz} = -\int_{\frac{3r_1}{2}}^{2r_1} \vec{E}\cdot d\vec{l}$$

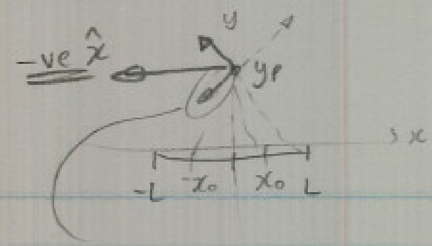
$$= -\frac{r_1\sigma_s}{\varepsilon_0}\int_{\frac{3r_1}{2}}^{2r_1} \frac{\hat{r}}{r} = \frac{-r_1\sigma_s}{\varepsilon_0}\left(\ln 2r_1 - \ln \frac{3r_1}{2}\right)$$

$$= \frac{-r_1\sigma_s}{\varepsilon_0}\ln\left(\frac{2r_1}{\left(\frac{3r_1}{2}\right)}\right) = \frac{-r_1\sigma_s}{\varepsilon_0}\ln\left(\frac{4}{3}\right)$$

$$|V_2| = \frac{r_1\sigma_s}{\varepsilon_0}\ln\left(\frac{4}{3}\right)$$

Q1.





$\underline{\underline{-x_0}}$

$\dfrac{\lambda_0 x}{2L}$

$-L < x < L$

$y = 0$

$z = 0$

$$\lambda_L(x) = \lambda_0 \frac{x}{2L} \qquad \vec{E}(0, y_p, 0) = ?$$

$$dE = \frac{dq_s}{4\pi\varepsilon_0} \cdot \frac{(\vec{r}_p - \vec{r}_s)}{(\vec{r}_p - \vec{r}_s)^{3/2}}$$

$$\vec{r}_p = y_p \hat{y}$$

$$\vec{r}_s = x_s \hat{x} + \cancel{y_s}^{\,0} \hat{y} + \cancel{z_s}^{\,0} \hat{z}$$

$$\vec{r}_p - \vec{r}_s = -x_s \hat{x} + (\cancel{y_p} - y_s)\hat{y} + \cancel{z_s}^{\,z_p} \hat{z}$$

$$\vec{r}_p - \vec{r}_s = -x_s \hat{x} - y_s \hat{y} + z_p \hat{z}$$

$$dq_s = \left(\frac{\lambda_0 x}{2L}\right) dx_s$$

$$dE = \frac{\lambda_0 x \, dx_s}{2L(4\pi\varepsilon_0)} \int \frac{-x_s\hat{x} - \cancel{y_s \hat{y}} + z_p \hat{z}}{(x_s^2 + \cancel{y_s^2} + z_p^2)^{3/2}}$$

$E = E_x + E_y + E_z \Rightarrow E_x = E_y = 0$ due to symmetry.

$$E = E_z = \frac{\lambda_0 z_p}{8L\pi\varepsilon_0} \int_{-L}^{L} \frac{x_s \, dx_s}{(x_s^2 + z_p^2)^{3/2}}$$

$$x_s = z_p \tan\theta$$

$$dx_s = z_p \sec^2\theta = \frac{z_p \, d\theta}{\cos^2\theta}$$

$$x_s^2 = z_p^2 \tan^2\theta$$

$$\frac{\sin\theta}{\cos\theta} = \tan\theta$$

$$\frac{2}{2} - \frac{3}{2} = -\frac{1}{2}$$

Integral becomes:

$$\int \sqrt{y_s^2 + z_p^2}\,\tan\theta \; \frac{\sqrt{y_s^2+z_p^2}}{\cos^2\theta}\,d\theta \cdot \frac{1}{\left[(y_s^2+z_p^2)(\sec^2\theta)\right]^{3/2}}$$

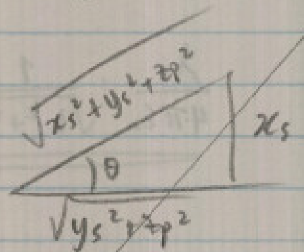
$$\int (y_s^2+z_p^2)\,\tan\theta\,d\theta \;\; \frac{1}{(y_s^2+z_p^2)^{3/2}} \;\; \frac{\cos^{\cancel{3}}\theta}{\cancel{\cos^2\theta}}$$

$$\int \frac{(y_s^2+z_p^2)}{(y_s^2+z_p^2)^{3/2}}\,\tan\theta\cos\theta\,d\theta$$

$$\left.\begin{matrix}\sin\theta \\ \cos\theta\end{matrix}\right\} \frac{d}{d\theta}$$

$$\int \frac{\sin\theta\,d\theta}{(y_s^2+z_p^2)^{1/2}} = \frac{-\cos\theta}{(y_s^2+z_p^2)^{1/2}}$$

$c^{A}H$



$$-\cos\theta = \frac{-(y_s^2+z_p^2)^{1/2}}{\sqrt{x_s^2+y_s^2+z_p^2}}$$

$$\left.-\frac{(y_s^2+z_p^2)^{1/2}}{\sqrt{x_s^2+y_s^2+z_p^2}}\right|_{-L}^{L} = -(y_s^2+z_p^2)^{1/2}\left[\frac{1}{L^2+\cancel{y_s^2}+z_p^2} - \right.$$

$\uparrow$
x_s